

# Symmetry & Controllability for Spin Networks with a Single-Node Control

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**Abstract**—We consider the relation of symmetries and subspace controllability for spin- $\frac{1}{2}$  networks with XXZ couplings, subject to perturbation of a single node by a local potential ( $Z$ -control). The Hamiltonians for such networks decompose into excitation subspaces. Focusing on the single excitation subspace, it is shown for single-node  $Z$ -controls that external symmetries are characterized by eigenstates of  $H_0$  which have zero overlap with the control node, and there are no internal symmetries. It is further shown that there are symmetries which persist even in the presence of random perturbations. For XXZ chains with uniform coupling strengths, a characterization of all possible symmetries is given which shows a strong dependence on the position of the node we control. We then show for Heisenberg and XX chains with uniform coupling strength subject to single-node  $Z$ -control that the lack of symmetry is both necessary and sufficient for subspace controllability. Finally, the latter approach is generalized to establish controllability results for simple branched networks.

**Index Terms**—non-linear control, controllability, symmetries, quantum systems, spin networks

## I. INTRODUCTION

In the study of quantum communication and quantum computation, spin chains and general spin networks are simple but very useful models to approximate physical systems [1]–[4], including molecules in NMR experiments [5] and ultracold atoms in optical lattices [6]. In the latter case it has been demonstrated experimentally that single and two-qubit gates can be implemented by controlling the external magnetic field, thus realizing universal quantum computation (QC) [7]–[9]. Universality is the capability of generating all unitary computational gates, which in the language of control theory is equivalent to system controllability, i.e., the capability to generate the Lie algebra of the unitary (or special unitary) group using both the system Hamiltonian and the available control Hamiltonians [10], [11]. Controllability is an important criterion for assessing a system’s capability for quantum information processing and many other applications of quantum control [12]–[19]. In principle, controllability can be verified by explicitly computing the dimension of the Lie algebra, but such computations quickly become intractable as the dimension of the system increases. Therefore, alternative controllability conditions are needed, and many of these have been studied and derived for spin networks of various couplings and topologies [14], [20], [21]. Most of the early results characterize conditions for full controllability on the whole Hilbert space, and often a surprisingly small number of

controls is necessary. In recent work it was shown that two independent local controls acting on the first spin suffice for full controllability for spin chains with coupling of Heisenberg type [22], while chains with XX-coupling control of at least the first two qubits is required [23], [24].

Alternatively, we would like to know what interesting tasks we can still perform given certain limited Hamiltonian resources. These questions are practically important, because in many real physical systems there are limitations on the Hamiltonian that can be implemented, and the quantum dynamics may be restricted to a subspace of the full Hilbert space [24], [26]–[28]. In this article, we study this subspace controllability problem. Specifically, for any spin- $\frac{1}{2}$  network with coupling of XXZ-type subject to  $Z$ -directional magnetic control fields, the Hilbert space of the system always decomposes into so-called excitation subspaces, which remain invariant under the dynamical evolution [27]. Furthermore, unlike in previous work on restricted local control, which has usually focused on control of first the spin, we explicitly study the effect of the position of the controlled node on the controllability on the single excitation subspace. We find a surprising relationship between actuator placement, controllability and dynamical *symmetries* of the system.

Symmetry is an important concept and a powerful tool in physics [29]. Dynamical symmetries provide an alternative perspective to the controllability problem, implying non-controllability, and are of interest as characterizing symmetries is usually easier than calculating the dynamical Lie algebra. This has been explored in several works on controllability of spin chains and networks [30]–[33]. In our case we are able to fully characterize all possible dynamical symmetries for a general XXZ-chain with local  $Z$ -control of a single spin in the single excitation subspace with the aid of the Bethe ansatz [34]. While dynamical symmetries imply that the system is not controllable, lack of symmetry does not always imply controllability. This raises the question under which conditions the converse holds, i.e., lack of symmetry implies controllability [31]. We show that for common models such as XX and Heisenberg chains and some branched networks lack of symmetry is not only a necessary but also a sufficient condition for controllability.

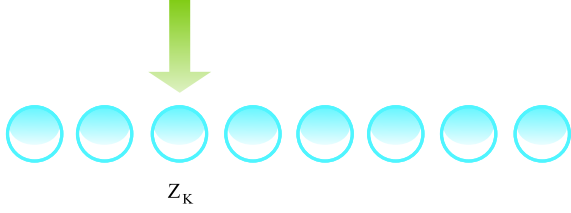


Fig. 1: XXZ chain with a local Z control  $H_1 = Z_k$ .

## II. XXZ NETWORKS WITH SINGLE-NODE CONTROL

**Intrinsic Hamiltonian:** The Hamiltonian of a network of  $N$  spin- $\frac{1}{2}$  particles with XXZ coupling is of the form

$$H_s = \frac{1}{2} \sum_{1 \leq m < n \leq N} \gamma_{mn} (X_m X_n + Y_m Y_n + \kappa Z_m Z_n), \quad (1)$$

where  $X, Y, Z$  are the standard Pauli operators and  $X_n$  denotes the  $N$ -fold tensor product whose  $n$ th factor is  $X$ , all others being the identity  $\mathbb{I}$ . The dimensionless constant  $\kappa$  determines the type of coupling such as Heisenberg ( $\kappa = 1$ ), XX ( $\kappa = 0$ ) or dipole ( $\kappa = -1$ ) coupling. The constants  $\gamma_{mn}$  determine the coupling strengths between nodes  $m$  and  $n$  in the network. Special cases of interest are chains with *nearest-neighbor* coupling, for which  $\gamma_{mn} = 0$  except when  $m = n \pm 1$ . A network is *uniform* if all non-zero couplings are equal, i.e.,  $\gamma_{mn} \in \{0, \gamma\}$ . To every spin network we can associate a simple graph representation with vertices  $\{1, \dots, N\}$  determined by the spins and edges by non-zero couplings, i.e., there is an edge connecting nodes  $m$  and  $n$  exactly if  $\gamma_{mn} \neq 0$ .

**Node Controls:** In this article our main interest are spin networks subject to a magnetic field in the  $z$ -direction or equivalently a  $Z$ -control acting *locally*, on a single spin (node/vertex) in the network as shown in Fig. 1 for a chain. The corresponding control Hamiltonian is

$$H_c = Z_k. \quad (2)$$

Neglecting dissipation, the dynamical evolution of the network subject to a time-varying control field  $f(t)$ , representing the magnitude of the magnetic field applied to node  $k$ , is governed by the Schrödinger equation

$$\dot{\rho} = \frac{i}{\hbar} [H_s + f(t)H_c, \rho]. \quad (3)$$

**Symmetries:** Following the definitions given in [33], a controlled spin network with Hamiltonian  $H_s + f(t)H_c$  is said to have an *external* or *commutation symmetry* if there is a Hermitian matrix  $S$  which commutes with both Hamiltonians, i.e.,  $[H_s, S] = [H_c, S] = 0$ . The existence of such a symmetry operator implies that the Hamiltonians  $H_s$  and  $H_c$  can be simultaneously block-diagonalized, and the Hilbert space decomposed into smaller invariant subspaces corresponding to the eigenspaces of the symmetry operator  $S$ .

The system Hamiltonian of an XXZ spin network always commutes with the total spin operator  $S_F = \sum_n (Z_n + I)/2$ ,  $[H_s, S_F] = 0$ , as does any control Hamiltonian of the form  $Z_k$ , and thus any XXZ spin network with only controls of this type always has this external symmetry. It is easy to see that

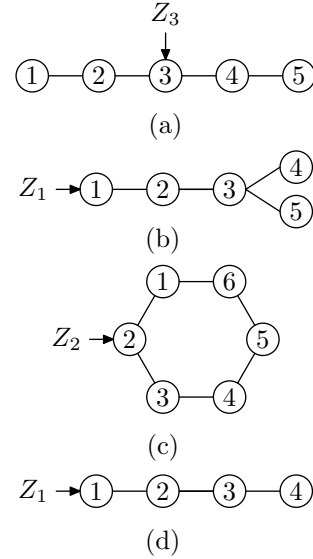


Fig. 2: Different spin networks with a local Z control at positions indicated by the arrows. (a),(b),(c) have permutation symmetry, (d) does not.

$S_F$  has  $N + 1$  distinct eigenvalues, ranging from  $n = 0$  to  $n = N$ , corresponding to the possible different numbers of excitations in the network, and hence we have  $N + 1$  invariant subspaces.

Another common type of external symmetry is *permutation symmetry*. A non-identity member of the permutation group  $\Pi$  is a graph symmetry of the *controlled network* if both  $H_s$  and  $H_c$  are invariant under the permutation  $\Pi$  of the spins. For a single-node control this means in particular that the permutation must fix the controlled node. The permutation  $\Pi$  induces a permutation operator  $P$  which commutes with both the intrinsic and control Hamiltonians and hence defines an external symmetry. In Fig. 2 there are permutation symmetries in (a), (b) and (c) but not in (d). Permutation symmetries commute with the total spin operator  $S_F$  and therefore induce symmetries on *all* excitation subspaces.

The existence of external symmetries immediately implies that the system is not controllable on the full Hilbert space  $\mathcal{H}$ , i.e., the dynamical Lie algebra  $\mathfrak{L}$  generated by  $iH_s$  and  $iH_c$  is a proper sub-algebra of  $\mathfrak{su}(2^N)$ <sup>1</sup>. This is easy to see as the existence of an external symmetry  $S$  means that the system and control Hamiltonians  $H_s$  and  $H_c$  are simultaneously blockdiagonalizable and therefore can not generate  $\mathfrak{su}(N)$  or  $\mathfrak{u}(N)$ . Specifically, if the Hilbert space decomposes,  $\mathcal{H} = \oplus_{d=1}^D \mathcal{H}_d$ , where  $\mathcal{H}_d$  are invariant subspaces, then the dynamical Lie algebra generated must be a subalgebra of  $\oplus_{d=1}^D \mathfrak{u}(\dim \mathcal{H}_d)$ . The system may still be controllable on one or more of the subspace  $\mathcal{H}_d$  of  $\mathcal{H}$ , however, if the dynamical Lie algebra on the subspace  $\mathcal{H}_d$  is equal to  $\mathfrak{u}(\dim \mathcal{H}_d)$  or  $\mathfrak{su}(\dim \mathcal{H}_d)$ . This is the notion of controllability we are interested in here, in particular the controllability of the system on the single excitation subspace  $\mathcal{H}_1$  with respect to the total excitation

<sup>1</sup>The set of unitary gates that can be implemented for a dynamical Lie algebra  $\mathfrak{L}$  is  $e^{\mathfrak{L}}$  and the system is controllable on the entire Hilbert space if  $\mathfrak{L} = \mathfrak{u}(2^N)$  or  $\mathfrak{L} = \mathfrak{su}(2^N)$  [11].

(symmetry) operator  $S_F$ .

### III. SUBSPACE SYMMETRIES AND CONTROLLABILITY

Denoting the spin- $\frac{1}{2}$  excitation basis vectors by  $|\uparrow\rangle$  and  $|\downarrow\rangle$ , the single excitation subspace  $\mathcal{H}_1$  is spanned by the  $N$  basis vectors:  $|\downarrow\uparrow\cdots\uparrow\rangle, |\uparrow\downarrow\cdots\uparrow\rangle, \dots, |\uparrow\uparrow\cdots\downarrow\rangle$ , which we can denote by  $|1\rangle, \dots, |N\rangle$ , for simplicity. We can define a basis for the  $N \times N$  antisymmetric matrices as follows:

$$\begin{aligned} x_{j,k} &= i(|j\rangle\langle k| + |k\rangle\langle j|), \\ y_{j,k} &= -|j\rangle\langle k| + |k\rangle\langle j|, \\ z_k &= i|k\rangle\langle k|, \end{aligned}$$

with  $1 \leq j < k \leq N$ . Restricted to  $\mathcal{H}_1$ , the Hamiltonians  $H_s$  and  $H_c$  are given by the  $N \times N$  matrices

$$H_0 = -i \sum_{n=1}^{N-1} \gamma_n x_{n,n+1} - i \sum_{n=1}^N \mu_n z_n \quad (4a)$$

$$H_1 = -i(\mathbb{I} - 2z_k) \quad (4b)$$

where  $\gamma_n$  is shorthand for  $\gamma_{n,n+1}$  here, and the diagonal elements are  $\mu_n = \mu_0 - (\gamma_{n-1} + \gamma_n)\kappa$  with  $\mu_0 = \frac{\kappa}{2} \sum_{n=1}^{N-1} \gamma_n$ , setting  $\gamma_0 = \gamma_N = 0$  for convenience.

For the special case of a uniformly coupled chain,  $\gamma_n = 1$ , we obtain  $\mu_0 = (N-1)\frac{\kappa}{2}$ ,  $\mu_1 = \mu_N = (N-3)\frac{\kappa}{2}$  and  $\mu_n = (N-5)\frac{\kappa}{2}$  for  $1 < n < N$ , and thus up to a multiple of the identity we have

$$H_0 = \begin{pmatrix} \kappa & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 1 & 0 & 1 \\ 0 & \cdots & 0 & 1 & \kappa \end{pmatrix}, \quad (5)$$

$H_1$  is a diagonal matrix with ones everywhere except for the  $k$  diagonal element, which is  $-1$ . Subtracting again and changing sign we can equivalently take

$$H_1 = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}. \quad (6)$$

Note that the addition of multiples of identity matrix to  $H_s$  and  $H_c$  does not change their commutation relations with the original Hamiltonian, and hence the Lie algebra generated by  $H_0$  and  $H_1$  differs from the Lie algebra generated by the original Hamiltonians at most by the identity  $\mathbb{I}$ .

In the following, let  $\mathfrak{L}$  denote the dynamical Lie algebra generated by  $H_0$  and  $H_1$ . If  $\dim(\mathfrak{L}) = N^2$  or  $N^2 - 1$  then  $\mathfrak{L} = \mathfrak{u}(N)$  or  $\mathfrak{su}(N)$  and the system is controllable on  $\mathcal{H}_1$ . The generators  $x_{k,j}, y_{k,j}, z_k$  defined above form a natural basis for the Lie algebra  $\mathfrak{u}(N)$ , and if we can generate them from  $-iH_0 = \sum_{j=1}^{N-1} x_{j,j+1}$  and  $-iH_1 = z_k$ , then  $\mathfrak{L} = \mathfrak{u}(N)$  and the system is controllable. We shall also use the following

commutation relations:

$$\begin{aligned} [x_{jk}, z_k] &= y_{jk} \\ [y_{jk}, z_k] &= -x_{jk} \\ [x_{jk}, x_{k\ell}] &= y_{j\ell} \\ [x_{jk}, y_{k\ell}] &= -x_{j\ell} \\ [x_{jk}, y_{jk}] &= 2(z_j - z_k) \end{aligned}$$

Moreover, to show controllability it suffices to show that we can generate all  $x_{n,n+1}$  and  $y_{n,n+1}$  for  $n = 1, \dots, N-1$ .

As noted above, permutation symmetries commute with the total spin operator  $S_F$  and therefore induce symmetries on all eigenspaces of  $S_F$ , including  $\mathcal{H}_1$ . Thus, for the XXZ networks shown in Fig. 2 we can immediately conclude that the system is not controllable on any excitation subspace (except the trivial ones  $n = 0$  and  $n = N$ ) if (a) we control the middle spin, (b) we control any spin outside the two equal-length branches, and (c) we control any spin on the ring. However, permutation symmetries are not the only possible symmetries, especially on  $\mathcal{H}_1$ .

**Example:** For an XX chain ( $\kappa = 0$ ) of length  $N = 7$  and  $k = 2$ , we find that both  $H_0$  and  $H_1$ , restricted to the single excitation subspace, commute with

$$M = \begin{pmatrix} 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & 0 \end{pmatrix},$$

which is not a permutation symmetry. We can indeed verify that the dimension of the Lie algebra is  $\dim(\mathfrak{L}) = 36 < 7^2$ , i.e., the system is not controllable on  $\mathcal{H}_1$ , and the restrictions of  $H_0$  and  $H_1$  to  $\mathcal{H}_1$  can be simultaneously block-diagonalized.

Firstly, we can now show generally that for systems with a single-node  $Z$ -type control, the external symmetries can be very easily characterized.

**Theorem 1.** *An XXZ spin network with a single node control  $Z_k$  has external symmetries on  $\mathcal{H}_1$  if and only if  $H_0$  has one or more eigenvectors  $|v\rangle$  with zero overlap with node  $|k\rangle$ , i.e.,  $\langle k|v\rangle = 0$ .*

*Proof:* Without loss of generality, we can choose a basis such that  $H_0$  is diagonal. From Favard's theorem [38] for tridiagonal matrices, we know that the eigenvalues of  $H_0$  are distinct. In such a basis  $H_1$  takes the form  $B_{k\ell} = \langle v_k|1\rangle\langle 1|v_\ell\rangle$  and setting  $C = [M, B]$  gives  $C_{k\ell} = (m_k - m_\ell)B_{k\ell}$ , which shows that  $C_{k\ell}$  can vanish only if  $B_{k\ell} = 0$  or  $m_k = m_\ell$ . If all  $B_{k\ell} \neq 0$  then  $C_{k\ell} = 0$  for all  $k, \ell$  is only possible if  $m_k = m_\ell$  for all  $k, \ell$ , i.e., if  $M$  is a multiple of the identity, in which case there is no symmetry. Thus a symmetry exists if and only if  $B_{k\ell} = 0$  for some  $k, \ell$ , which is equivalent to  $\langle v_k|1\rangle = 0$  for some  $k$ , i.e., the existence of an eigenvector  $v_k$  that has zero overlap with the controlled node. ■

[25] found a similar characterization in the setting of control by relaxation, and [33] showed the existence of external

symmetries to be equivalent to the existence of eigenstates of the system Hamiltonian that have no overlap with the pendant vertex for spin networks with a pendant-type control.

We shall refer to systems with no external or commutation symmetries as *indecomposable*. Note that spin networks given by coupling graphs with multiple disjoint components are always decomposable as are systems with permutation symmetries, but as we have seen, these are by no means the only external symmetries the system might have. External symmetries imply that the Hilbert space can be written as a direct sum of subspaces that are invariant under the dynamics. Assuming we have found all external symmetries and decomposed the Hilbert space into invariant subspaces that are not further reducible, the dynamics on each invariant subspaces can be subject to *internal Lie group symmetries*, characterized by the existence of a *unitary* or *anti-unitary* operator  $S$  such that

$$(iH_0)^T S + S(iH_0) = (iH_1)^T S + S(iH_1) = 0, \quad (7)$$

where  $H_0$  and  $H_1$  are required to be trace-zero and  $A^T$  denotes the transpose of  $A$ . These internal symmetries can be divided into orthogonal and symplectic symmetries, and the existence of such a symmetry implies that the dynamical Lie algebra  $\mathfrak{L}$  generated by  $H_0$  and  $H_1$  is a subalgebra of  $\mathfrak{so}(N)$  or  $\mathfrak{sp}(N)$ , respectively, and thus the system is again not controllable on the respective subspace. For spin networks with a particular single controlled coupling (a *pendant control*) it was recently found that internal symmetries on the single excitation subspace  $\mathcal{H}_1$  are of orthogonal type and related to the existence of a bipartite structure of the network [33]. In contrast, internal symmetries never occur for XXZ spin networks with single-node  $Z$ -controls.

**Theorem 2.** *An indecomposable XXZ network with a single-node  $Z$ -control  $-iH_1 = z_k$  does not permit internal symmetries on  $\mathcal{H}_1$ .*

*Proof:* Let  $\bar{H}_m = H_m - \frac{1}{N} \text{Tr}(H_m)H_m \mathbb{I}_N$  for  $m = 0, 1$ , be the zero-trace versions of the Hamiltonians  $H_0$  and  $H_1$ . We need to show there is no internal symmetry between  $\bar{H}_0$  and  $\bar{H}_1$ . As our Hamiltonians are both real-symmetric the internal symmetry condition can be simplified

$$\bar{H}_0 S + S \bar{H}_0 = 0, \quad \bar{H}_1 S + S \bar{H}_1 = 0.$$

We see that when  $\bar{H}_1$  is diagonal,  $\bar{H}_1 = \text{diag}(a_n)$ , the latter condition is equivalent to  $(a_m + a_n)S_{mn} = 0$ , which implies  $S_{mn} = 0$  unless  $a_m = -a_n$ . In particular, if  $H_1 = z_k$  then we have  $a_k = 1$  and  $a_n = -\frac{1}{N-1}$  for  $n \neq k$  and thus the sum of any two diagonal elements of  $\bar{H}_1$  never vanishes. ■

Thus, remarkably for this type of control, we only need to consider external symmetries, which we shall refer to simply as symmetries in the following. In fact, the proof of the previous theorem shows that a decomposable spin network with  $Z$ -controls admits internal symmetries on the single excitation subspace if only if  $N$  is even and we collectively control exactly half of all nodes, in which case  $\bar{H}_1$  will contain equal numbers of  $+1$  and  $-1$  entries, which can cancel.

**Theorem 3.** *An XXZ chain with arbitrary coupling strengths with a single  $Z$  control at the end node is controllable on the single-excitation subspace  $\mathcal{H}_1$ .*

*Proof:* To show controllability we calculate the dynamical Lie algebra  $\mathfrak{L}$  generated by  $-iH_0$  and  $-iH_1$ , noting that the Hamiltonians take the form (4) and (6) with  $\gamma_n \neq 0$  for any  $1 \leq n < N$  and  $k = 1$ . The commutators

$$\begin{aligned} \gamma_1^{-1}[-iH_0, z_1] &= y_{1,2} \\ [z_1, y_{1,2}] &= x_{1,2}, \\ z_1 + \frac{1}{2}[x_{1,2}, y_{1,2}] &= z_2 \end{aligned}$$

immediately give us the three generators of the Lie algebra. Using these generators we can define a reduced system generated by  $-iH_2$  and  $z_2$  with

$$-iH_2 = -iH_0 - x_{1,2} - \mu_1 z_1 = \sum_{k=2}^{N-1} \gamma_k x_{k,k+1} + \sum_{k=2}^N \mu_k z_k,$$

which represents an XXZ chain of length  $N-1$ . By the same procedure as above, we can now generate  $x_{2,3}$ ,  $y_{2,3}$  and  $z_3$ ; iterating the procedure  $N-2$  times, we can generate  $x_{n,n+1}$ ,  $y_{n,n+1}$  and  $z_n$  for  $n = 1, \dots, N-1$ , i.e., all the generators corresponding to the simple roots of the Lie algebra  $\mathfrak{u}(N)$ . ■

Controllability implies the non-existence of symmetries but we can see directly that there are no external symmetries. Theorem 1 shows that the first entry of any eigenvector of  $H_0$  cannot be zero. Suppose we have a eigenvector  $v$  of  $H_0$  with  $v_1 = 0$ . Then  $H_0 v = \lambda v$  gives

$$\begin{pmatrix} \mu_1 & \gamma_1 & 0 & \cdots & 0 \\ \gamma_1 & \mu_2 & \gamma_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \gamma_{N-2} & \mu_{N-1} & \gamma_{N-1} \\ 0 & \cdots & 0 & \gamma_{N-1} & \mu_N \end{pmatrix} \begin{pmatrix} 0 \\ v_2 \\ \vdots \\ v_{N-1} \\ v_N \end{pmatrix} = \begin{pmatrix} \gamma_1 v_2 \\ \gamma_2 v_3 \\ \vdots \\ \vdots \end{pmatrix},$$

which in turn implies  $v_2 = v_3 = \dots = 0$ , i.e.  $v = 0$ . Thus any eigenvector of  $H_0$  cannot be zero at its first entry, i.e., there is no external symmetry.

This end-controllability result can be generalized to the case where we collectively control  $k$  spins from the end of the chain, i.e.  $-iH_1 = \sum_{j=1}^{k \leq N} z_j$ .

**Theorem 4.** *For XXZ chains of length  $N$  with  $Z$ -control of  $k < N$  spins at one end of the chain, the system is controllable and hence has no symmetry on  $\mathcal{H}_1$ .*

*Proof:* Given Theorem 3, we only need to show that from  $-iH_0$  and  $-iH_1 = z_1 + \dots + z_k$ ,  $k < N$ , we can generate  $z_1$ , and then the controllability follows. We have

$$\begin{aligned} [-iH_1, -iH_0] &\rightarrow y_{k,k+1} \\ [y_{k,k+1}, -iH_1] &\rightarrow x_{k,k+1} \\ [x_{k,k+1}, y_{k,k+1}] &\rightarrow z_k - z_{k+1} \end{aligned}$$

and defining

$$-iH'_0 \equiv -iH_0 - \gamma_k x_{k,k+1} + \frac{1}{2}(\mu_k - \mu_{k+1})(z_k - z_{k-1})$$

we further have

$$\begin{aligned} [-iH'_0, y_{k,k+1}] &\rightarrow -x_{k-1,k+1} + x_{k,k+2} \\ [-iH'_0, x_{k,k+1}] &\rightarrow -y_{k-1,k+1} + y_{k,k+2} \\ [-x_{k-1,k+1} + x_{k,k+2}, -y_{k-1,k+1} + y_{k,k+2}] \\ &\rightarrow (z_{k+2} - z_{k+1}) - (z_k - z_{k-1}) \end{aligned}$$

as well as

$$\begin{aligned} [-iH'_0, z_{k+1} - z_k] &\rightarrow y_{k+1,k+2} + y_{k-1,k} \\ [z_{k+1} - z_k, y_{k+1,k+2} + y_{k-1,k}] &\rightarrow x_{k+1,k+2} + x_{k-1,k} \\ [x_{k+1,k+2} + x_{k-1,k}, y_{k+1,k+2} + y_{k-1,k}] \\ &\rightarrow (z_{k+2} - z_{k+1}) + (z_k - z_{k-1}) \end{aligned}$$

Thus, we can generate  $z_{k+2} - z_{k+1}$  and  $z_k - z_{k-1}$ .

Moreover, from  $[z_k - z_{k-1}, -iH'_0]$  we can get  $y_{k-1,k}$  and we get  $x_{k-1,k}$ . Hence, we find the following rule: starting from  $z_{k+1} - z_k$ ,  $x_{k,k+1}$  and  $y_{k,k+1}$ , we can generate  $z_{k+2} - z_{k+1}$  and  $z_k - z_{k-1}$ . Then, starting from  $z_k - z_{k-1}$ ,  $x_{k-1,k}$ , and  $y_{k-1,k}$ , we can analogously generate  $z_{k-1} - z_{k-2}$ ,  $x_{k-2,k-1}$ , and  $y_{k-2,k-1}$ . Repeating this process, we can sequentially generate  $z_k - z_{k-1}$ ,  $z_{k-1} - z_{k-2}$ ,  $\dots$ ,  $z_2 - z_1$ , and together with  $-iH_1$ , we can finally generate  $z_1$ , and the controllability follows. ■

Based on these results one might conjecture that we always have subspace controllability, at least for inhomogeneous chains, regardless of which node we control. But this is not the case, and in fact some symmetries are extremely robust even in the presence of inhomogeneity.

**Theorem 5.** *An XX chain of odd length always has an external symmetry (and is thus not controllable) on  $\mathcal{H}_1$  if the index  $k$  of the controlled spin is even.*

*Proof:* The Hamiltonian of XX chain reduces to

$$-iH_0 = \sum_{n=1}^{N-1} \gamma_n x_{n,n+1}$$

Favard's theorem [38] for tridiagonal matrices guarantees that  $H_0$  has  $N$  distinct eigenvalues  $\lambda_j$  with corresponding eigenvectors  $v_j = (v_{kj})$  satisfying

$$v_{kj} = (-1)^{k-1} \frac{f_{k-1}(\lambda_j)}{\gamma_1 \cdots \gamma_{k-1}},$$

where the polynomial  $f_{n+1}(\lambda)$  is determined by the recurrence relation:

$$f_{n+1}(\lambda) = \lambda f_n(\lambda) - \gamma_n^2 f_{n-1}(\lambda),$$

for  $n > 0$  and  $f_0 = 1$  and  $f_{-1} = 0$ . Moreover,  $f_N(\lambda)$  is the characteristic polynomial of  $H_0$ . If  $N$  is odd then  $f_N$  is an odd function too, so  $\lambda = 0$  is a root of  $f_N(\lambda)$  with eigenvector  $v_0 = (f_0(0), f_1(0), \dots)$  and  $f_1(0) = f_3(0) = \dots = f_N(0) = 0$ , i.e. all even entries vanish. Hence if  $k$  is even, there is an external symmetry and the system is not controllable. ■

A general XXZ chain, either homogeneous or not, is called centro-symmetric with respect to the centre of the chain if the couplings  $\gamma_n := \gamma_{n,n+1}$  between adjacent spins satisfy  $\gamma_n = \gamma_{N-1-n}$  for  $n = 1, \dots, N-1$ .

**Theorem 6.** *A centro-symmetric XXZ chain is not controllable on the  $\mathcal{H}_1$  subspace if  $N = 2k - 1$ , where  $k$  is the controlled spin.*

*Proof:* For a centro-symmetric chain of length  $N = 2k - 1$  the characteristic polynomial is of the form  $f_N(\lambda) = f_{k-1}(\lambda)p(\lambda)$ , where both  $f_{k-1}(\lambda)$  and  $p(\lambda)$  are polynomials of degree  $k - 1$  and  $k$ , respectively. This shows that  $k - 1$  of the roots  $\lambda_m$  of  $f_N(\lambda)$  must be roots of  $f_{k-1}(\lambda)$ , and hence  $f_{k-1}(\lambda_m) = 0$  for these  $\lambda_m$ , showing that the corresponding eigenvectors  $|\lambda_m\rangle$  are “dark states,” i.e., satisfy  $[H_S + f(t)H_c]|\lambda_m\rangle = 0|\lambda_m\rangle$  for any control  $f(t)$ , if we control the middle spin  $k = (N + 1)/2$ . Thus the maximum controllable subspace is has dimension  $k + 1$ . ■

#### IV. CHARACTERIZATION OF SYMMETRIES FOR HOMOGENEOUS XXZ CHAINS

From Theorem 1,  $H_0$  and  $H_1$  have an external symmetry if and only if there exists some eigenvector  $v_j = (v_{kj})$  of  $H_0$  such that  $v_{kj} = 0$ . For a homogeneous chain the Hamiltonian  $H_0$  is tridiagonal with uniform values for the off-diagonal elements and zeros on the diagonal except for the first and last entry, and if we define

$$v_0 = \kappa v_1, \quad v_{N+1} = \kappa v_N. \quad (8)$$

then the eigenvalue equation  $H_0 v = E v$  can be written as  $v_{k-1} + v_{k+1} = E v_k$  for  $k = 1, \dots, N$ . This suggests that the eigenvectors  $v$  are of the form (Bethe ansatz [37])

$$v_k = A z^k + B z^{-k} \quad (9)$$

with  $z = e^{i\theta}$ . Substituting into (8) gives:

$$A + B = \kappa(Az + Bz^{-1}) \quad (10a)$$

$$Az^{N+1} + Bz^{-(N+1)} = \kappa(Az^N + Bz^{-N}). \quad (10b)$$

The first equation gives  $B = Az \frac{1-\kappa z}{\kappa-z}$ , assuming  $A \neq 0$ ,  $z \neq 1$ . Inserting this into the second equation we obtain

$$z^{2N} = \frac{(1-\kappa z)^2}{(\kappa-z)^2} \Rightarrow z^N = \pm \frac{1-\kappa z}{\kappa-z}. \quad (11)$$

The condition for the existence of a symmetry,  $v_k = 0$ , thus becomes  $Az^{2k} + B = 0$ , and inserting  $B = Az \frac{1-\kappa z}{\kappa-z}$ ,

$$z^{2k-1} = -\frac{1-\kappa z}{\kappa-z} = \pm z^N. \quad (12)$$

Equation (12) allows us to find all possible values of  $\kappa$  that allow external symmetries for given  $N$  and  $k$ . If  $N = 2k - 1$ , for instance, then any  $\kappa$  satisfies (12) and the system always has a symmetry. Indeed, this is the case of an odd chain where we control the central spin, and it is easy to see that this system has a permutation symmetry as discussed before. Similarly, if  $N = 2k$  then there are no symmetries for any  $\kappa$  as  $z^N = \pm z^{N-1}$  can be satisfied only if  $z = 0, \pm 1$  but these solutions do not correspond to valid Bethe eigenvectors.

More generally, substituting  $z = e^{i\theta}$  into (12) gives

$$e^{i\phi} \equiv e^{i(2k-1)\theta} = -\frac{1-\kappa z}{\kappa-z} = \frac{(\kappa \cos \theta - 1) + i\kappa \sin \theta}{(\kappa - \cos \theta) - i \sin \theta}. \quad (13)$$

This equation contains two independent equations for the real and the imaginary parts:

$$\begin{aligned}\cos \phi (\kappa - \cos \theta) + \sin \phi \sin \theta &= \kappa \cos \theta - 1 \\ \sin \phi (\kappa - \cos \theta) - \sin \theta \cos \phi &= \kappa \sin \theta.\end{aligned}$$

The solutions of these two equations are, respectively,

$$\begin{aligned}\kappa &= \frac{\cos(\theta + \phi) - 1}{\cos \phi - \cos \theta}, & \cos \phi &\neq \cos \theta \\ \kappa &= \frac{\sin(\phi + \theta)}{\sin \phi - \sin \theta}, & \sin \phi &\neq \sin \theta.\end{aligned}$$

Substituting  $\phi = (2k-1)\theta$  and using elementary trigonometric identities, both of these solutions simplify to

$$\kappa = \frac{\sin(k\theta)}{\sin(k-1)\theta}. \quad (14)$$

Substituting  $z = e^{i\theta}$  into Eq. (12) shows further that we must have  $e^{i(2k-1)\theta} = \pm e^{iN\theta}$ , or  $e^{i(N-2k+1)\theta} = \pm 1$ . For  $2k-1 < N$  this gives

$$\theta = \frac{j\pi}{N - (2k-1)}, \quad j \in \mathbb{Z}. \quad (15)$$

Thus, for given  $N$  and  $k$ , from (15) we can find the corresponding  $\theta$  and  $\kappa$  such that the system has a symmetry. We can traverse all possible values of  $N$ ,  $k$  and  $j$  in order to find all types of homogeneous XXZ chain that has an external symmetry. A set of these values are summarized in Table I for small values of  $N$ . An immediate consequence of this is:

**Theorem 7.** *There is at most a countably-infinite number of  $\kappa$  that permit external symmetry.*

The theorem implies that for a generic  $\kappa$  uniformly chosen at random from the real line, there will be no symmetry for any  $N$  and  $k$ . We must be careful, however, with results such as this because the most common values for  $\kappa$  for real physical systems are  $\kappa = 0$  (XX-coupling),  $\kappa = 1$  (Heisenberg), and  $\kappa = -1$  (dipole coupling) and for these special values of  $\kappa$  symmetries exist for many choices of  $N$  and  $k$ .

For  $\kappa = 0$  (**XX-coupling**) Eq. (11) gives  $z^{N+1} = \pm 1$  with  $z = e^{i\theta}$  or  $\theta = \frac{j\pi}{N+1}$ , Eq. (10a) gives  $A = -B$  and (9) thus becomes

$$v_{kj} = Ae^{ik\theta_j} - Ae^{-ik\theta_j} = C \sin(k\theta_j) \quad (16)$$

for  $j, k = 1, \dots, N$  with corresponding eigenvalues are  $E_j = 2 \cos(\theta_j)$ . A simple calculation reveals the normalization constant to be  $C = \sqrt{2/(N+1)}$ . For the system to have an external symmetry the  $k$ th entry of one of the eigenvectors  $v_j$  must vanish. In this case this happens only if  $\sin(k\theta_j) = 0$ , or  $k\theta_j = \ell\pi$ , i.e., if there exists an integer  $\ell > 0$  such that  $kj = (N+1)\ell$ , or equivalently if and only if  $N+1$  and  $k$  have a common divisor  $> 1$  or  $\gcd(N+1, k) = g > 1$ , where  $\gcd(\cdot, \cdot)$  represents the greatest common divisor. Hence, the Hamiltonians have an external symmetry if and only if  $\gcd(N+1, k) = g > 1$ .

Similarly, for  $\kappa = 1$  (**Heisenberg coupling**) Eq. (11) gives  $z^{2N} = 1$  and we obtain [36]:

$$v_{kj} = \cos((2k-1)\theta_j), \quad \theta_j = \frac{j\pi}{2N}, \quad (17)$$

$N$	$k$	$\theta = \frac{j\pi}{N-(2k-1)}$	$\kappa$
5	2	$\frac{\pi}{2}$	0
6	2	$\frac{\pi}{3}, \frac{2\pi}{3}$	$\pm 1$
6	3		No solution
7	2	$\frac{\pi}{2}$	0
7	3		No solution
8	2	$\frac{\pi}{2}, \frac{2\pi}{5}$	$\pm 2 \cos \frac{\pi}{5}$
8	3	$\frac{\pi}{3}$	0
8	4		No solution
9	2	$\frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}$	$\pm\sqrt{3}, \pm 1, 0$
9	3	$\frac{\pi}{4}, \frac{3\pi}{4}$	$\pm \frac{\sqrt{2}}{2}$
9	4	$\frac{\pi}{2}$	0
10	2	$\frac{j\pi}{7}, j = 1, \dots, 6$	$\pm 2 \cos \frac{\pi}{5}, \pm 2 \cos \frac{2\pi}{5}, \pm 2 \cos \frac{3\pi}{5}$
10	3	$\frac{\pi}{5}, \frac{2\pi}{5}$	$\pm 1$
10	4		No solution
10	5		No solution

TABLE I: For different values of  $N$  and  $k$ ,  $\theta$  and  $\kappa$  which make  $v_{jk} = 0$  can be calculated.

which shows that the system has an external symmetry if and only if  $\gcd(N, 2k-1) = g > 1$ . This is identical to the result for  $\kappa = -1$  (**dipole-coupling**)<sup>2</sup>. This is true is general, owing to the fact there is a 1-to-1 correspondence between the eigenvalues and eigenvectors of XXZ chains with opposite  $\kappa$ .

**Theorem 8.** *Homogeneous XXZ chains with opposite values of  $\kappa$  have the same external symmetries.*

*Proof:* Let the Hamiltonian of an XXZ chain be  $H_0 = H_0[\kappa]$ . Assuming it satisfies the eigenvalue equation:  $H_0 v = E v$ , and defining  $P = \prod_{j=1}^{N/2} Z_{2j}$ , we have  $(P H_0 P^\dagger) P v = E P v$ , and  $P H_0 P^\dagger = -(X X + Y Y) + \kappa Z Z = -H_0[-\kappa]$ . Hence, if  $E$  and  $v$  are the eigenvalue and the eigenvector of  $H_0[\kappa]$ , then  $-E$  and  $P v$  are the eigenvalue and the eigenvector of  $H_0[-\kappa]$ . Moreover, as  $P v = [v_1, -v_2, v_3, -v_4, \dots]^T$ , if  $N$  and  $k$  are chosen such that  $v_k = 0$  then the  $k$ th entry of  $P v$  is also equal to zero. Hence,  $H_0[\kappa]$  and  $H_0[-\kappa]$  have the same external symmetries for given  $N$  and  $k$ . ■

## V. CHARACTERIZATION OF $\mathcal{H}_1$ -CONTROLLABILITY FOR UNIFORM XX AND HEISENBERG CHAINS

In the previous section we characterized the symmetries for XXZ chains, and we know that the absence of symmetry is a necessary condition for controllability. Unfortunately, it is not sufficient in general.

**Example.** Consider an XX-spin network composed of  $N = 10$  spins as illustrated in FIG. 3 with a  $Z$ -control applied (jointly) to several spin nodes: (a)  $H_1 = z_1$ , (b)  $H_1 = z_1 + z_2$ , (c)  $H_1 = z_1 + z_2 + z_3$  and (d)  $H_1 = z_1 + z_2 + z_3 + z_4$ . Calculating the dynamical Lie algebra  $\mathcal{L}$  generated by  $iH_0$  and  $iH_1$ , we have in (a) and (b)  $\dim(\mathcal{L}) = 81$ ; in (c)  $\dim(\mathcal{L}) = 100$ ; in (d)  $\dim(\mathcal{L}) = 25$ , i.e., only in case (c) do we have  $\mathcal{H}_1$ -subspace controllability. The result in the first two cases is due to the existence of an external symmetry, in this case a single dark state, i.e., an eigenstate  $v$  of  $H_0$  that has no overlap with the controlled spin,  $\langle k | v \rangle = 0$ , and a controllable subspace of dimension 9. In case (d), however, one can verify

<sup>2</sup>This result applies to a linear chain with nearest-neighbor coupling, which is somewhat artificial for a dipole-coupled chain as dipole coupling tends to be long-range.

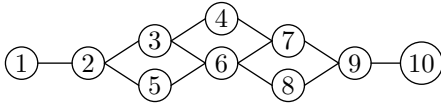


Fig. 3: An XX spin network with  $N = 10$  spins. On  $\mathcal{H}_1$ , we consider multi-node Z control, with (a)  $H_1 = z_1$ , (b)  $H_1 = z_1 + z_2$ , (c)  $H_1 = z_1 + z_2 + z_3$  and (d)  $H_1 = z_1 + z_2 + z_3 + z_4$ .

that no symmetries exist, and the Lie algebra generated is an irreducible representation of  $\mathfrak{u}(5)$ . This result can be explained if we realize that the first excitation subspace Hamiltonians  $H_0$  and  $H_1$  for this network are in fact identical to the second excitation subspace Hamiltonians for a uniform linear chain of length  $N = 5$ , and the Lie algebra for this system is indeed  $\mathfrak{u}(5)$  with the second excitation subspace corresponding to the 10-dimensional anti-symmetric (irreducible) representation of  $\mathfrak{u}(5)$ .

**Example.** The above correspondence between the graph in FIG. 3 and the second excitation subspace Hamiltonian of an XX chain of length  $N = 5$  also holds for the inhomogeneous case. For example, the single excitation subspace Hamiltonians for a 10-spin network

$$H_0 = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 4 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 4 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \end{pmatrix}$$

with collective Z-control of the first four spins,  $H_1 = z_1 + z_2 + z_3 + z_4$ , is equivalent to the second excitation subspace Hamiltonian of a chain of length  $N = 5$  where the couplings between adjacent spins are  $\gamma_1 : \gamma_2 : \gamma_3 : \gamma_4 = 1 : 2 : 3 : 4$  and we control the first spin only, and we find  $\dim(\mathfrak{L}) = 25 < 10^2$ , or  $\mathfrak{L} = \mathfrak{u}(5)$  and no symmetries either. This shows that even networks with non-uniform coupling without any symmetry can be non-controllable.

From the above examples, we see that there are XXZ spin networks without any symmetries in  $\mathcal{H}_1$  that are nonetheless not controllable on this subspace. Nevertheless, we shall show that for a particular type of spin network, the linear chain with a single controlled node, lack of symmetry is not only necessary but also sufficient for controllability on  $\mathcal{H}_1$ . We shall give rigorous proofs for XX and Heisenberg chains, which are of most practical interest, but the same techniques could be applied to prove controllability for other types of chains.

#### A. XX Chain

**Theorem 9.** For an XX chain of length  $N$  with local control  $z_k$ , the system is controllable on  $\mathcal{H}_1$  if and only if  $\gcd(N + 1, k) = 1$ .

Note that this condition is equivalent to the no-symmetry condition derived in the previous section.

*Proof:* We already know that the system is controllable on the single excitation subspace for  $k = 1$ . Also, by symmetry, we only need to discuss the case when  $1 \leq k \leq N/2$ . Assuming  $1 < k \leq N/2$ , we would like to determine all the operators we can generate from  $H_0$  and  $H_1$  through commutation relations.

**Result 1.** For an XX chain on  $\mathcal{H}_1$  with  $H_1 = -iz_k$  and  $N = rk + m$ ,  $0 \leq m < k$ , we can generate  $z_{qk}$  with  $1 \leq q \leq r$ , starting from  $z_k$ .

We can generate the following elements:

$$\begin{aligned} [-iH_0, z_k] &\rightarrow y_{k-1,k} + y_{k+1,k} \\ [y_{k-1,k} + y_{k+1,k}, z_k] &\rightarrow x_{k-1,k} + x_{k+1,k} \\ -iH_2 &\equiv -iH_0 - (x_{k-1,k} + x_{k+1,k}) \\ [-iH_2, x_{k-1,k} + x_{k+1,k}] &\rightarrow y_{k-2,k} + y_{k+2,k} \\ [y_{k-2,k} + y_{k+2,k}, z_k] &\rightarrow x_{k-2,k} + x_{k+2,k} \\ &\dots\dots\dots \\ [-iH_2, x_{2k-2,k} + x_{2k,k}] &\rightarrow y_{1,k} + y_{2k-1,k} \\ [y_{1,k} + y_{2k-1,k}, z_k] &\rightarrow x_{1,k} + x_{2k-1,k} \\ [-iH_2, x_{1,k} + x_{2k-1,k}] &\rightarrow y_{2k,k} \\ [y_{2k,k}, z_k] &\rightarrow x_{2k,k} \\ [y_{2k,k}, x_{2k,k}] &\rightarrow z_{2k} \end{aligned}$$

Thus, starting from  $-iH_0$  and  $z_k$ , we can generate  $z_{2k}$ , and continuing this process, we can sequentially generate  $z_{3k}, z_{4k}, \dots$

For  $N = rk$  this implies that we can generate  $z_N$ , which is equivalent to  $z_1$  and implies controllability by Theorem 3.

**Result 2.** If we can generate  $z_{k_1}$  and  $z_{k_2}$  with  $k_1 - k_2 = 1$ , starting with  $z_k$  and  $H_0$ , then the system is controllable.

If we can generate two local operators  $z_{k_1}$  and  $z_{k_2}$  with  $k_1 - k_2 = 1$  from  $z_k$  and  $H_0$  then we can further generate

$$H'_0 = x_{1,2} + \dots + x_{k_1-2,k_1-1} + x_{k_1+1,k_1+2} + \dots + x_{N-1,N},$$

and together with  $z_{k_1-1}$  we can sequentially generate  $z_{k_1-2}, z_{k_1-3}, \dots$  until we obtain  $z_1$ , analogous to the proof of Theorem 3, and Theorem 3 itself then again implies controllability.

**Result 3.** Let  $N = rk + m$ ,  $0 \leq m < k$ . If  $r = 2r'$  is even, then  $z_k$  and  $H_0$  can generate  $z_m$ ; If  $r = 2r' + 1$  is odd, then  $z_k$  and  $H_0$  can generate  $z_{k-m}$ .

With the above results we can apply the Euclidean algorithm to complete the proof of Theorem 9.

Let  $n_1 = k$ . For  $\gcd(N + 1, n_1) = 1$ , and  $n_1 \mid (N + 1)$ , we have  $N + 1 = r_1 n_1 + m_2$ ,  $0 < m_2 < n_1$ . By Result 3, starting with  $z_{n_1}$ , we can generate  $z_{n_2}$ , where  $n_2 = m_2$  or  $n_2 = n_1 - m_2$ , depending upon the parity of  $r_1$ . we also have  $\gcd(n_1, n_2) = 1$  and thus  $n_1 = r_2 n_2 + n_3$ , with  $n_3 < n_2$  and  $\gcd(n_2, n_3) = 1$ . Thus, from the Euclidean algorithm, we can generate the sequence  $n_1, n_2, \dots, n_{f-1}, n_f$  with  $n_{f-1} - n_f = 1$ . Corresponding to such a sequence on the chain, we can generate the operator pairs  $(z_{p_k}, z_{p'_k})$ , with  $z_{p_1} = z_1$ ,  $z_{p'_1} = z_{n_1}$ ,  $z_{p_2} = z_{(r_2-1)n_2}$ , and  $z_{p'_2} = z_{r_2 n_2}$ , with  $p_2 - p'_2 = n_1$  and  $p'_1 - p'_2 = n_3$ . Then we can generate  $z_{p_3}$ , with  $p_3 = p'_2 - r_3 n_3$

and  $p'_3 = p'_2 - (r_3 - 1)n_3$ . We have  $p_3 - p_2 = n_4$  and  $p_3 - p'_3 = n_3$ . We can then generate  $p_4$  and  $p'_4$  between  $p_3$  and  $p'_3$  with  $p'_3 - p'_4 = n_3$  and  $p'_4 - p_4 = n_4$ . We repeat this process until we get  $p_f$  and  $p'_f$  where one of the two is the neighbor of  $p_{f-1}$  or  $p'_{f-1}$ . Thus, we can generate two operators  $z_a$  and  $z_b$  such that  $a - b = 1$ , and by Result 2, we derive the controllability.

If  $\gcd(N + 1, n_1) = g > 1$ , then analogous to the above argument, we can similarly generate  $n_2, n_3, \dots, n_f$  with  $n_f = g$ , and similarly the pair  $(z_{p_j}, z_{p'_j})$ , with  $p_j - p'_j = n_j$ ,  $j = 1, \dots, f$ , and  $n_1 > n_2 > \dots > n_f = g$ , and then we can generate  $z_g$ . Thus all operators  $z_\ell$  generated from  $z_k$  and  $H_0$  satisfy  $\ell = sg$ , and the operators  $z_m$  with  $g \nmid m$  can not be individually generated, so the dynamical Lie algebra generated from  $z_k$  and  $H_0$  is strictly smaller than  $U(N)$ , and the system is not controllable. ■

Next, we try to demonstrate the above constructive proof through examples. For  $N = 11$  and  $n_1 = k = 5$ , we have

$$\begin{aligned} N + 1 = 12 &= 2 \times 5 + 2 = r_1 n_1 + m_2 \\ n_1 = 5 &= 2 \times 2 + 1 = r_2 n_2 + n_3 \end{aligned}$$

According to Results 1 and 3, we can sequentially generate  $z_5, z_{10}$ , then  $z_2, z_4$ , and then  $z_1$ . Hence we have controllability by Theorem 3. However, if  $N = 14$  and  $k = 5$  then it is easy to see that the only  $z_\ell$  operators we can generate from  $-iH_0$  and  $z_5$ , are  $z_5$  and  $z_{10}$ , and the system is not controllable in this case.

It is worth noting that from the above proof, not only have we proved the lack of symmetry is equivalent to controllability on  $\mathcal{H}_1$  for a uniform XX chain, but we have also derived the dynamical Lie algebra  $\mathcal{L}$  when the system is not controllable. Moreover, the proof implies a result which is also useful when discussing the Heisenberg chain in the following section:

**Result 4.** *Given that we can generate  $z_{p_i}$ ,  $i = 1, 2, 3$ , with  $p_1 < p_2 < p_3$  and  $\gcd(p_2 - p_1, p_3 - p_1) = 1$ , we can further generate two neighboring  $z_\ell$  and  $z_{\ell+1}$  with  $p_1 \leq \ell < \ell + 1 \leq p_3$ . Hence, the system is controllable.*

## B. Heisenberg Chain

Analogous to XX chain, the correspondence between the lack of symmetry and the controllability is summarized in the following theorem:

**Theorem 10.** *For a Heisenberg chain of length  $N$  with uniform coupling strengths and a local Z control  $z_k$ , the system is controllable on  $\mathcal{H}_1$  if and only if  $\gcd(N, 2k - 1) = 1$ .*

Again the condition for controllability is equivalent to the no-symmetry condition. First of all, we investigate the new operators generated by  $-iH_0$  and  $-iH_1$ . Assuming  $N > 2k$  and for  $k > 1$ , similar to the XX chain case, we have

$$\begin{aligned} [-iH_0, z_k] &\rightarrow y_{k-1,k} + y_{k+1,k} \\ [y_{k-1,k} + y_{k+1,k}, z_k] &\rightarrow x_{k-1,k} + x_{k+1,k} \\ -iH_2 &\equiv -iH_0 - (x_{k-1,k} + x_{k+1,k}) \\ [-iH_2, x_{k-1,k} + x_{k+1,k}] &\rightarrow y_{k-2,k} + y_{k+2,k} \\ [y_{k-2,k} + y_{k+2,k}, z_k] &\rightarrow x_{k-2,k} + x_{k+2,k} \\ &\dots\dots\dots \\ [-iH_2, x_{2,k} + x_{2k-2,k}] &\rightarrow y_{1,k} + y_{2k-1,k} \\ [y_{1,k} + y_{2k-1,k}, z_k] &\rightarrow x_{1,k} + x_{2k-1,k} \\ [-iH_2, x_{1,k} + x_{2k-1,k}] &\rightarrow y_{1,k} + y_{2k,k} \\ [y_{1,k} + y_{2k,k}, z_k] &\rightarrow x_{1,k} + x_{2k,k} \\ [-iH_2, x_{1,k} + x_{2k,k}] &\rightarrow x_{2,k} + x_{2k+1,k} \\ &\dots\dots\dots \\ [-iH_2, x_{k-2,k} + x_{3k-3,k}] &\rightarrow x_{k-1,k} + x_{3k-2,k} \\ [-iH_2, x_{k-1,k} + x_{3k-2,k}] &\rightarrow x_{3k-1,k} \\ [x_{3k-1,k}, z_k] &\rightarrow y_{3k-1,k} \\ [x_{3k-1,k}, y_{3k-1,k}] &\rightarrow z_{3k-1} \end{aligned}$$

Thus, from  $z_k$ , we can sequentially generate  $x_{k-1,k} + x_{k+1,k}, \dots, x_{k-m,k} + x_{k+m,k}$ , until  $x_{1,k} + x_{2k-1,k}$ . When we generate some  $x$  operator (e.g.  $x_{1,k} + x_{2k-1,k}$ ), we can always generate the corresponding  $y$  operator (e.g.  $y_{1,k} + y_{2k-1,k}$ ) and vice versa, so in the following, we only concentrate on the  $x_{i,j}$  operators that can be generated. Different from XX model, and due to the component  $z_1$  in  $-iH_2$ , the next operator we can generate is  $x_1 + x_{2k,k}$ , rather than  $x_{2k,k}$  as in the XX chain case. Then we can sequentially generate  $x_{j,k} + x_{j+2k-1,k}$ , and finally we get  $x_{k+(2k-1),k}$ , and hence  $z_{k+(2k-1)}$ . Analogously, starting from  $z_{k+(2k-1)}$ , we can sequentially generate  $x_{k+(2k-1)-m,k} + x_{k+(2k-1)+m,k}$ ,  $m = 1, \dots, 2k - 2$ , and finally we get  $x_{k+2(2k-1),k}$  and hence  $z_{k+2(2k-1)}$ . Continuing such process, we have

**Result 5.** *For Heisenberg chain, from  $z_k$  and  $H_0$ , we can sequentially generate  $z_{k+m(2k-1)}$ , with  $1 \leq m \leq r$  and  $k + r(2k - 1) < N$ . In particular, if  $N = k + r(2k - 1)$ , then the system is controllable.*

Next, similar to the XX chain case, we have the following result

**Result 6.** *For Heisenberg chain, if from  $H_0$  and  $H_1$  we can generate two neighboring  $z_{k_1}$  and  $z_{k_2}$  with  $k_1 - k_2 = 1$ , then the system is controllable.*

Now we are ready to prove Theorem 10. Assuming  $r(2k - 1) < N < (r + 1)(2k - 1)$ , for convenience of analysis, we construct a modified model  $M_2$  (FIG. 4), corresponding to the original Heisenberg chain,  $M_1$  in (5).

If  $r(2k - 1) < N < k + r(2k - 1)$ , then in  $M_2$ , we extend the original chain to length  $N' = k + r(2k - 1)$ , and add new XX couplings in  $H_0$  between the nearest neighbors  $x_{j,j+1}$ ,  $N \leq j \leq N' - 1$ , but delete the term  $z_N$ . Thus, in the modified system Hamiltonian  $H_0$  the right end of  $M_2$  is of XX-type interaction, rather than Heisenberg-type. As a result, in the modified model, starting from  $z_k$ , we can generate  $z_j$



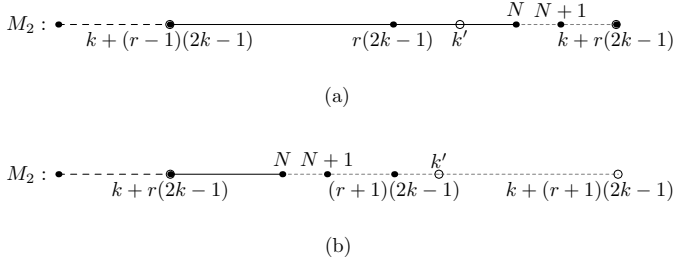


Fig. 4: Heisenberg chain  $M_1$  is modified into the model  $M_2$ , by extending the chain length to (a)  $N' = k + r(2k - 1)$ , if  $r(2k - 1) < N < k + r(2k - 1)$ ; (b)  $N' = k + (r + 1)(2k - 1)$ , if  $k + r(2k - 1) \geq N < (r + 1)(2k - 1)$ . In both cases, we can generate  $z_{k'}$  with  $k' = 2N + 1 - (k + r(2k - 1))$ .

with  $j = k + (r - 1)(2k - 1)$ , and from  $z_j$ , we can generate  $x_{2j-N,k} + x_{N,k}$ , the same as in the original Heisenberg model. Then continuing the calculation, in the modified model  $M_2$ , we then generate  $x_{2j-N-1,j} + x_{N+1,j}$  (corresponding to  $x_{2j-N-1,j} + x_{N,j}$  in  $M_1$ ), then  $x_{2j-N-2,j} + x_{N+2,j}$ , and finally  $x_{k+r(2k-1),j}$  and  $z_{k+r(2k-1)}$ , which corresponds to the generated operator  $z_{k'}$  in  $M_1$ , with  $k' = 2N + 1 - (k + r(2k - 1)) > k$  (FIG. 4 (a)). If we make the following identification: node  $N + 1$  in  $M_2$  corresponding to node  $N$  in  $M_1$ , and node  $N + 2$  in  $M_2$  corresponding to node  $N - 1$  in  $M_1$ , and so forth, then the operators generated from  $z_j$  in  $M_2$  are 1-to-1 corresponding to the  $z$  operators generated in  $M_1$ . Moreover, we can see that all these corresponding pairs of nodes are mirror symmetric with respect to the middle point between nodes  $N$  and  $N + 1$ . Hence, once a new operator is generated in  $M_2$ , we can always get the corresponding new operator generated in  $M_1$  through the mirror symmetry. For instance, if we can generate two neighboring operators  $z_\ell$  and  $z_{\ell+1}$  in  $M_2$ , they must correspond to two neighboring  $z$  operators in  $M_1$  as well.

If  $k + r(2k - 1) < N < (r + 1)(2k - 1)$ , then in  $M_2$  we extend the spin length to  $k + (r + 1)(2k - 1)$ . Since we have generated  $z_{k+r(2k-1)}$  in  $M_1$ , by mirror symmetry, it corresponds to the operator  $z_{k'}$  which can be generated in  $M_2$ , with  $k' = 2N + 1 - (k + r(2k - 1)) > k$  (FIG. 4 (b)).

To complete the proof of the main theorem, let  $r(2k - 1) \leq N < (r + 1)(2k - 1)$ . From the previous discussion, we only need to discuss what new operators can be generated for the modified model  $M_2$ , and by the 1-to-1 correspondence between  $M_1$  and  $M_2$ , we can recover what new operators can be generated for  $M_1$ . In both of the two cases  $r(2k - 1) < N < k + r(2k - 1)$  and  $k + r(2k - 1) < N < (r + 1)(2k - 1)$ , we can always generate  $z_{k'}$  with  $k' = 2N + 1 - (k + r(2k - 1)) > k$ , satisfying

$$k' - k = 2N - (r + 1)(2k - 1).$$

If  $\gcd(N, 2k - 1) = 1$ , we have  $\gcd(k' - k, 2k - 1) = 1$ . If  $r(2k - 1) < N < (r + 1)(2k - 1)$ , we have  $p_1 \equiv k + (r - 1)(2k - 1) < k' < k + r(2k - 1) \equiv p_3$ ; if  $k + r(2k - 1) < N < (r + 1)(2k - 1)$ , we have  $p_1 \equiv k + r(2k - 1) < k' < k + (r + 1)(2k - 1) \equiv p_3$ . But in both cases,  $p_3 - p_1 = 2k - 1$

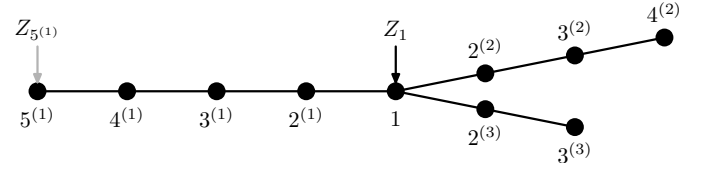


Fig. 5: Star shape spin networks with three subchains, satisfying subchain lengths are pairwise coprime. We find that in both cases (i)  $H_1 = z_1$ , and (ii)  $H_1 = z_{5^{(1)}}$  the system is controllable.

and  $(p_3 - p_1, k' - p_1) = 1$ . Hence, the conditions of Result 4 are satisfied, and we can generate two neighboring operators  $z_{\ell'}$  and  $z_{\ell'+1}$  with  $p_1 \leq \ell' < \ell' + 1 \leq p_3$  in  $M_2$ , which must correspond to two neighboring  $z$  operators in  $M_1$ . Hence the original system  $M_1$  is controllable.

If  $\gcd(N, 2k - 1) = g > 1$ , then by the same reasoning for XX chain, the only  $z_j$  operator we can generate are  $z_j = z_{mg}$ , i.e.,  $j$  is multiple of  $g$ . And all the other  $z_j$  operator are coupled with other operators. Hence we cannot generate all  $z_j$  and the dynamical Lie algebra is strictly smaller than  $U(N)$ .

The same arguments can be applied to show that lack of symmetry is also the necessary and sufficient condition for controllability on  $\mathcal{H}_1$  for  $\kappa = -1$ , and other cases could be studied similarly. So far we have fully characterized the 1-to-1 correspondence between the external symmetries of  $H_0$  and  $H_1$ , and the non-controllable cases of the system.

## VI. BRANCHED NETWORKS

In previous sections we have studied symmetries and controllability for the simplest type of spin network, i.e., the linear chain. The same techniques can be applied to more complex spin networks.

Besides the spin chain, the next simplest model of a spin network is the star shape branched network, with a central spin, connected with  $m$  number of spin subchains. For  $m = 2$  this is equivalent to the chain case. The first non-trivial case is  $m = 3$ , i.e., the T-shape spin network, as illustrated in FIG. 5. For a uniformly-coupled XX-type branched network, the central spin can be indexed as 1 and the  $m$  number of branches indexed as  $2^{(p)}, \dots, \ell_p^{(p)}$ ,  $p = 1, \dots, m$ . On  $\mathcal{H}_1$ , the branched network Hamiltonian and the local control  $Z$  Hamiltonian on node  $k^{(q)}$  are written as:

$$H_0 = \sum_{p=1}^m \left( x_{1,2^{(p)}} + \sum_{j^{(p)}} x_{j^{(p)}, j^{(p)}+1} \right)$$

$$H_1 = z_{k^{(q)}}$$

We start with the simplest symmetry to identify, permutation symmetry. If the local control is on one of the three subchains, say,  $H_1 = z_{k^{(1)}}$ , and the other two branches are of the same length,  $\ell_2^{(2)} = \ell_3^{(3)}$ , then there is a permutation symmetry, and the system is not controllable. In the following, we shall discuss cases of the position of spin  $k^{(q)}$ .

**Case 1:** We control the far end of one branch (for example,  $H_1 = z_{5^{(1)}}$  in FIG. 5).

N	$(\ell_1^{(1)}, \ell_2^{(2)}, \dots, \ell_m^{(m)})$	Symmetry	$\dim(\mathcal{L})$	Controllable
10	(5,4,3)	No	100	Yes
11	(6,4,3)	Yes	65	No
12	(6,5,3)	Yes	101	No
12	(7,4,3)	No	144	Yes
13	(6,5,4)	Yes	144	No
14	(7,5,3,2)	No	196	Yes

TABLE II: Symmetry and controllability in  $\mathcal{H}_1$  for different XX branched networks.

N	$(\ell_1^{(1)}, \ell_2^{(2)}, \dots, \ell_m^{(m)})$	Symmetry	$\dim(\mathcal{L})$	Controllable
8	(2,3,5)	Yes	50	No
9	(2,4,5)	Yes	65	No
9	(2,3,6)	No	$9^2$	Yes
10	(3,4,5)	Yes	65	No
10	(2,4,6)	No	$10^2$	Yes
10	(2,3,7)	No	$10^2$	Yes
11	(3,4,6)	No	$11^2$	Yes
12	(2,4,8)	Yes	65	No
12	(3,5,6)	No	$12^2$	Yes
13	(4,5,6)	No	$13^2$	Yes
14	(4,5,7)	No	$14^2$	Yes
14	(2,4,10)	No	$14^2$	Yes
15	(2,4,11)	Yes	122	No

TABLE III: Symmetries and controllability for Heisenberg branched networks in  $\mathcal{H}_1$ .

We assume  $H_1 = z_{\ell_1^{(1)}}$ . Analogous to the discussion of XX chain, we can sequentially generate  $z_{\ell_1-1^{(1)}}, \dots, z_{2^{(1)}}, z_1$  as well as the associated  $x_{j,j+1}$  and  $y_{j,j+1}$ . Thus we can generate  $z_1$  and the XX chain Hamiltonian formed by the other two subchains.

According to Theorem 9, we have if  $\gcd(\ell_2^{(2)}, \ell_3^{(3)}) = 1$ , then the system is controllable; otherwise, it is not controllable.

**Case 2:** We control the central spin  $z_1$  (for example,  $H_1 = Z_1$  in FIG. 5).

If two of the subchains are the same length, then the system is not controllable. If the chains are of different length, then motivated by the results in Theorem 9, we have the following conjecture:

**Conjecture 1.** *For star shape XX-type networks with three subchains, and local Z control on the central spin 1, there is no external symmetry and the system is controllable if and only  $\gcd(\ell_j^{(j)}, \ell_k^{(k)}) = 1$  for  $j \neq k$ .*

For small values of  $N$ , we have calculated the symmetry and the dynamical Lie algebra  $\mathcal{L}$  for all different patterns of such branched network, with a few examples illustrated in Table II, all satisfying Conjecture 1.

For Heisenberg-type of branched networks, we can similarly calculate the symmetry and  $\mathcal{L}$  for controllability, with a few examples illustrated in Table III. However, the explicit relationship between the values of the parameters  $(N, \ell_1^{(1)}, \ell_2^{(2)}, \dots, \ell_m^{(m)})$  and the patterns of branched networks with symmetry requires further investigation in the future.

## VII. CONCLUSION

In this work we have studied symmetries and controllability of XXZ spin networks subject to local Z-controls on the single excitation subspace. External symmetries for such systems can

be easily characterized: such symmetries exist if and only if there are eigenstates of the system Hamiltonian that have no overlap with the control node. If no such symmetries exist then the system is indecomposable. Unlike systems where we control the coupling between two spins (or an edge in the associated graph of the spin network), indecomposable systems which have Z-controls applied to one or more spins have internal Lie algebra symmetries only in very exceptional cases.

For linear XXZ chains we have further characterized all possible values of  $\kappa$  that allow the system to have external symmetries on the single excitation subspace. We find that there are at most countably many values of  $\kappa$  which permit any external symmetries, i.e. for a generic  $\kappa$  there will be no symmetries, and we expect the system to be controllable for any local Z control. However, for the values of  $\kappa$  that are most relevant in real physical systems:  $\kappa = 0$  for XX coupling,  $\kappa = 1$  for Heisenberg coupling and  $\kappa = -1$  for dipole coupling, there are symmetries in many cases. The existence or absence of symmetries depends on the position of the controlled node in the chain relative to the length of the chain and the type of coupling in a very peculiar manner. This shows that the choice of controlled node — or the placement of the “actuator” in control terminology — is very significant.

For chains with non-uniform coupling strengths, we find that the inhomogeneity usually breaks symmetries which are present for the uniformly-coupled case with the same topology, as one might expect. Surprisingly however, there are certain symmetries which are robust even in the presence of inhomogeneities. Absence of symmetries is a necessary condition for controllability. In many cases it also appears to be a sufficient condition but there are examples of systems without symmetries that are not controllable. We have shown that for uniform XX and Heisenberg chains, lack of symmetry is the necessary and sufficient condition for the system’s controllability on  $\mathcal{H}_1$ . Finally, we have shown how to apply the techniques used to establish controllability for spin chains to more complex networks such as branched networks. We have characterized the possible symmetries and propose a conjecture of the controllability condition.

Similar techniques could be applied to discuss the more complex spin networks, as well as the relationship between symmetry, controllability and actuator placement in other excitation subspaces. For instance, it could be applied to explain the observation that an antiferromagnetic chain with local end-spin control appears to be controllable in the largest excitation subspace [27].

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## REFERENCES

- [1] S. C. Benjamin and S. Bose, Phys. Rev. Lett. **90**, 247901 (2003)

- [2] S. Lloyd, A. J. Landahl and J.-J. E. Slotine, Phys. Rev. A **69**, 012305 (2004)
- [3] A. Kay and M. Ericsson, New J. Phys. **7**, 143 (2005)
- [4] K. G. H. Vollbrecht and J. I. Cirac, Phys. Rev. Lett. **100**, 010501 (2008)
- [5] L. M. K. Vandersypen and I. L. Chuang, Rev. Mod. Phys. **76**, 1037 (2005)
- [6] J. Garcia-Ripoll and J. Cirac, New J. Phys. **5**, 76 (2003)
- [7] M. A. Nielsen and I. L. Chuang, Quantum computation and quantum information (Cambridge University Press, Cambridge, 2000)
- [8] D. P. DiVincenzo, Phys. Rev. A **51**, 1015 (1995)
- [9] A. Barenco et al., Phys. Rev. A **52**, 3457 (1995)
- [10] V. Jurdjevic, H. Sussmann, J. Diff. Equations **12**, 313 (1972)
- [11] D. D'Alessandro, *Introduction to Quantum Control and Dynamics* (Chapman & Hall/CRC, Boca Raton, 2008)
- [12] C. Altafini, Controllability of quantum mechanical systems by root space decomposition of  $su(n)$ , J. Math. Phys., **43**, 2051-2062 (2002)
- [13] H. Fu, S. G. Schirmer, A. I. Solomon, Complete controllability of finite-level quantum systems, J. Phys. A **34**, 1679-1693 (2001)
- [14] F. Albertini and D. D'Alessandro, The Lie algebra structure and controllability of spin systems, Linear Algebra and its Applications, **350**, 213-235 (2002)
- [15] F. Albertini and D. D'Alessandro, Notions of controllability for bilinear multilevel quantum systems, IEEE Transactions on Automatic Control **48**, No. 8, 1399-1403 (2003)
- [16] S. G. Schirmer, J. V. Leahy, A. I. Solomon, Degrees of controllability for quantum systems and application to atomic systems, J. Phys. A **35**, 4125-4141 (2002)
- [17] S. G. Schirmer, I. C. H. Pullen, A. I. Solomon, Controllability of Quantum Systems, In: Hamiltonian and Lagrangian Methods in Nonlinear Control (ISBN 0-08-044278-1) 2003
- [18] S. G. Schirmer, I. C. H. Pullen, A. I. Solomon, Controllability of multipartite quantum systems and selective excitation of quantum dots, J. Opt. B **7**, S293-S299 (2005)
- [19] F. Albertini and D. D'Alessandro, Controllability of Quantum Walks on Graphs, arXiv:1006.2405 (2010)
- [20] N. Khaneja, Chem. Phys. **267**, 11 (2001).
- [21] D. Burgarth, S. Bose, C. Bruder, and V. Giovannetti, Phys. Rev. A **79**, 060305(R) (2009).
- [22] R. Heule, C. Bruder, D. Burgarth and V. M. Stojanovic, arXiv:1007.2572 (2010).
- [23] S. G. Schirmer, I. C. H. Pullen, and P. J. Pemberton-Ross, Phys. Rev. A **78**, 062339 (2008)
- [24] A. Kay and P. J. Pemberton-Ross, Phys. Rev. A **81**, 010301(R) (2010)
- [25] D. Burgarth, Phys. Rev. Lett. **99**, 100501 (2007)
- [26] D. Burgarth, et al., Phys. Rev. A **81**, 040303(R) (2010).
- [27] X. Wang, A. Bayat, S. G. Schirmer and S. Bose, Phys. Rev. A **81**, 032312 (2010).
- [28] R. Heule, C. Bruder, D. Burgarth and V. M. Stojanovic, arXiv:1010.5715 (2010)
- [29] W. Greiner and B. Muller, *Quantum Mechanics: Symmetries* (Springer, Berlin, 1989).
- [30] T. Polack, H. Suchowski, and D. J. Tannor, Phys. Rev. A **79**, 053403 (2009)
- [31] U. Sander and T. Schulte-Herbrüggen, et al., arXiv:0904.4654 (2009).
- [32] R. Zeier and T. Schulte-Herbrüggen, arXiv:1012.5256 (2010)
- [33] P. J. Pemberton-Ross, A. Kay and S. G. Schirmer, Phys. Rev. A **82**, 042322 (2010)
- [34] H. Bethe, Zeitschrift für Physik A, **71**, 205 (1931)
- [35] A. Iserles, *A First Course in the Numerical Analysis of Differential Equations* (Cambridge University Press, 1996).
- [36] S. Bose, Phys. Rev. Lett. **91**, 207901 (2003).
- [37] M. Karabach and G. Müller, Comput. in Phys. **11**, 36 (1997).
- [38] T. S. Chihara, An Introduction to Orthogonal Polynomials, Routledge, London, 1978.